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Fundamental Valuation Of Patents in Continuous Time: A Note

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Abstract

In this note, we show how to solve for the fundamental (or bubble-free) value of a patent in continuous time using two methods: the method of integrating factor and the Laplace transform. Not only do these methods deliver a solution, they also provide conditions for when the solution is unique.

Keywords: Patents; Differential Equations; Integrating factor; Laplace transform; Bubbles

JEL Classification: 031; 040; C65

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1 Introduction

Since the 1980's, there has been a growing academic literature investigating the relationship between intellectual property rights (IPRs) and innovation (Romer, 1990; Grossman and Helpman, 1991; Aghion and Howitt, 1992). Harnessing R&D and innovation has been a priority for policymakers, particularly as technological progress is considered to be the key driver of economic growth in the long run (Solow, 1956).

The production of new ideas or designs generates new knowledge. New knowledge carries considerable economic value. However, it has features that make it problematic for the market system to handle properly (Arrow, 1962). Specifically, knowledge is a public good and public goods have two basic attributes. First, they are non-rival in consumption. Second, they are 'non-excludable' meaning that it is not possible to prevent others from enjoying it once available. IPRs in general address this problem by attacking the 'non-appropriability' of knowledge that lies at the heart of this market failure. Specifically, by rewarding innovators with property rights on their discoveries, patents and copyrights are legal mechanisms that attempt to bring the private benefits of invention closer in line with the social benefits.

When a new design or blueprint for a computer chip is discovered, the inventor receives a patent from the government. It is assumed that the patent lasts forever. The question is, what is the price of a patent for a new design? The answer (assuming no uncertainty and perfect foresight) is the present discounted value of the profits to be earned (Romer, 1990).

This note describes two different approaches to solve for the price of a patent in continuous time: the method of integrating factor and the Laplace transform. Not only do these methods deliver a solution, they also provide conditions for when the solution is unique.

The remainder of this article is organized as follows. Section II discusses the compensation that an innovator receives (patents) for incurring the fixed cost of R&D activity in Romer(1990). Sections III and IV explain how we solve for the price of a patent using, (i) the method of integrating factor, and (ii) the Laplace transform. Section V concludes.

2 The price of a patent in Romer (1990)

In order to motivate research, successful innovators have to be compensated in some manner. The basic problem is that the creation of a new design or blueprint is costly (R&D expenditure) but could then be used in a nonrival way by all potential users of the design. It would be efficient, ex post, to make the design freely available to all, but this practice fails to provide the ex ante incentives for further discoveries.

Romer (1990) in his seminal paper considers an institutional setup in which the inventor of a new design retains a perpetual monopoly right (enforced through explicit patent protection) over the production and sale of a capital goods that uses his or her design. The flow of monopoly profits provides the incentive for invention.

Therefore, the price of a patent (P_A) is the present value of profit flow i.e.,

$$P_A(t) = \int_t^{\infty} \pi(s) e^{-\int_t^s r(\tau)d\tau} ds \quad , \tag{2.1}$$

where $\pi(s)$ is the profit flow of date s and r is the risk-free interest rate. Romer (1990) exploits the Leibniz's rule for differentiation of a definite integral, to get[1]

$$r(t)P_A(t) = \pi(t) + \dot{P}_A(t) \quad . \tag{2.2}$$

The left-hand side of equation (2.2) is the interest earned from investing, P_A, in a bank; the right-hand side is the profits plus the capital gain or loss that results from the change in the price of the patent (Jones, 1995). Equation (2.2) is the no arbitrage condition. In fact, the absence of arbitrage opportunities is a fundamental principle underlying the modern theory of financial asset pricing.

The reasoning is as follows. Suppose an investor has 'x' dollars to invest for 1-year. He has two options. First, he can put the money in a bank for 1-year and earn the interest rate, r. Alternatively, he can purchase a patent for 1-year, earn the profits in that period, and then sell the patent. In equilibrium, it must be the case that the rate of return from both of these investments is the same. If not, everyone would jump at the more profitable investment, driving its return down.

Rewriting equation (2.2) yields,

$$r(t) = \frac{\pi(t)}{P_A(t)} + \frac{\dot{P}_A(t)}{P_A(t)} \quad . \tag{2.3}$$

---

[1] The Leibniz formula if the limits of integration are function of x and $f(x, y)$ is a function of two variables that can be integrated with respect to 't' and differentiated with respect to x:

$$F'(x) = \int_{a(x)}^{b(x)} f_x(x,t)dt + f(x, b(x))b'(x) - f(x, a(x))a'(x) \quad .$$

if $a(x) = a$ is a constant and $b(x) = x$, the rule simplifies to:

$$F'(x) = f(x,x) + \int_a^x f_x(x,t)dt \quad .$$

In steady-state, r is a constant in this model. Therefore, $\dot{\pi}(t)/\pi(t) = \dot{P}_A(t)/P_A(t)$. Furthermore, the rate of growth of profit equals the growth rate of population $(n > 0)$ (Jones, 1995).[2]

Finally, substituting, $\dot{P}_A(t)/P_A(t) = n$ in equation (2.3) above and rearranging yields the price of a patent along the balanced growth path i.e.,

$$P_A(t) = \frac{\pi(t)}{r - n} \quad . \tag{2.4}$$

This equation (the present discounted value of the profits to be earned) gives the price of a patent along the balanced growth path in these models. It tells us what influences the price of a patent in equilibrium. A higher r indicates that the present discounted value of profits is lower which naturally leads to a lower price for the patent.[3] In contrast, if n is higher this would encourage more R&D activity, as the scale of the economy (market size) is growing quickly, increasing demand for new products and potential profits. This in turn leads to a higher price for the patent.

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[2] In Romer (1990), population growth is assumed to be zero i.e., $n = 0$. Therefore, the price of a patent along the balanced growth path is given by, $P_A(t) = \left(\frac{1}{r}\right)\pi(t)$.

[3] An alternative way to understand this is to think of r as representing the return a potential investor could earn by putting their money in a bank for 1-year, rather than purchasing a patent.

3 Method of Integrating factor

3.1 Integrating factor

Integrating factors are useful for solving Ordinary differential equations (ODE) that can be expressed in the form,

$$y' + P(x)y = Q(x) \quad .$$

The basic idea is to find some 'unknown' function, say $M(x)$ called the "integrating factor", which we can multiply through our differential equation in order to bring the left-hand side under a common derivative. For the canonical first-order linear ODE shown above, the integrating factor is $e^{\int P(x)dx}$.

Let us begin by expressing the no arbitrage condition equation (2.2) in the standard form,

$$\dot{P}_A(t) - rP_A(t) = -\pi(t) \quad , \tag{3.1}$$

with initial condition, $P_A(0) = P_A(t_0)$. This is a first-order ODE with a variable forcing term.

Multiplying equation (3.1) throughout by the integrating factor, $M(t)$, yields

$$M(t)\dot{P}_A(t) - M(t)rP_A(t) = -M(t)\pi(t) \quad . \tag{3.2}$$

The left-hand side of equation (3.2) equals, $\frac{\mathrm{d}}{\mathrm{d}t}\Big[M(t)P_A(t)\Big]$, if $M(t) = e^{-\int r dt}$. Thus, equation (3.2) can be written as,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[M(t)P_A(t)\Big] = -M(t)\pi(t) \quad . \tag{3.3}$$

Integrating both sides with respect to 't' and re-arranging gives,

$$\begin{aligned} P_A(t) &= -\frac{1}{M(t)}\int \pi(t)M(t)dt - \frac{c_1}{M(t)} \\ &= -e^{\int r dt}\int \pi(t)e^{-\int r dt}dt - c_1 e^{\int r dt} \\ &= -e^{\int_{t_0}^{t} r d\tau}\int_{t_0}^{t} \pi(s)e^{-\int_{t_0}^{s} r d\tau}ds + ce^{\int_{t_0}^{t} r d\tau} \quad , \end{aligned} \tag{3.4}$$

where 'τ' and 's' are dummy variables and $c = -c_1$ (Barro and Sala-i-Martin, 2004).

Finally, setting $t = 0$ solves for the arbitrary constant, $c = P_A(t_0)$. Hence the general solution for this first-order ODE is,

$$P_A(t) = P_A(t_0)e^{\int_{t_0}^{t} r d\tau} - e^{\int_{t_0}^{t} r d\tau}\int_{t_0}^{t} \pi(s)e^{-\int_{t_0}^{s} r d\tau}ds \quad . \tag{3.5}$$

Multiply equation (3.5) throughout by, $e^{-\int_{t_0}^{t} r d\tau}$, and rearrange to get,

$$P_A(t_0) = \int_{t_0}^{t} \pi(s)e^{-\int_{t_0}^{s} r d\tau}ds + P_A(t)e^{-\int_{t_0}^{t} r d\tau} \quad . \tag{3.6}$$

The first term on the right-hand side is the fundamental value of the patent. The second term on the right-hand side is the difference between the market value of the patent, $P_A(t_0)$, and its fundamental value. By definition, this difference represents a bubble (or non-fundamental solution). Ruling out non-fundamental or extraneous solution i.e., letting $t \to \infty$, we get,

$$P_A(t_0) = \int_{t_0}^{\infty} \pi(s) e^{-\int_{t_0}^{s} r d\tau} ds + \lim_{t \to \infty} P_A(t) e^{-\int_{t_0}^{t} r d\tau} \quad . \tag{3.7}$$

Replacing t by T and t_0 by t in equation (3.7) to get,

$$P_A(t) = \int_{t}^{\infty} \pi(s) e^{-\int_{t}^{s} r d\tau} ds \quad . \tag{3.8}$$

Another way of describing our terminal condition would be as a 'side' or 'transversality' condition designed to rule out speculative bubbles. We can also see that this condition will both ensure uniqueness and rule out bubbles. Since profits grow at a constant rate, $n > 0$, in steady-state we have,

$$\begin{aligned} P_A(t) &= \int_{t}^{\infty} \pi(t) e^{ns} e^{-\int_{t}^{s} r d\tau} ds \\ &= \frac{\pi(t)}{r-n} \quad . \end{aligned} \tag{3.9}$$

Equation (3.9) is the value of the patent along the balanced growth path in Romer (1990).

4 Solving for the fundamental value of a patent using the Laplace transform

4.1 The transform

In this section we discuss and solve the no arbitrage condition using the Laplace transform. The Laplace transform is a very powerful method to solve certain types of ODE's and PDE's. The transform takes a differential equation and turn it into an algebraic equation. If the algebraic equation can be solved, applying the inverse transform gives us our desired solution.

Since the interest rate is a constant, r, and profits are growing at a constant rate, $n > 0$, we can write equation (2.2) as,

$$\dot{P}_A(t) - rP_A(t) = -\pi(t_0)e^{nt} \quad , \tag{4.1}$$

where $\pi(0) = \pi(t_0) > 0$, is the initial value of profit.[4] Moreover, it is convenient to express equation (4.1) in the standard form,

$$\dot{x}(t) - ax(t) = -\pi(t_0)e^{ct} \quad , \tag{4.2}$$

where $x(t) = P_A(t)$, $a = r$, $c = n$ and initial condition $x(0) = x(t_0)$.

[4]We know that the variable $\pi(t)$ grows at the constant rate n. Thus, $\pi(t)$ satisfies the differential equation $\frac{\dot{\pi}(t)}{\pi(t)} = n$ i.e., $\dot{\pi}(t) = n\pi(t)$. The solution for this homogeneous differential equation is, $\pi(t) = \pi(t_0)e^{nt}$.

Definition of the Laplace transform

Suppose that f is a real- or complex-valued function of the (time) variable $t > 0$ and s is a real or complex parameter. We define the Laplace transform of f as

$$\begin{aligned} F(s) = \mathcal{L}\big[f(t)\big] &= \int_0^\infty e^{-st} f(t)dt \\ &= \lim_{\tau\to\infty} \int_0^\tau e^{-st} f(t)dt \quad , \end{aligned} \tag{4.3}$$

whenever the limit exists (as a finite number). When it does, the integral (4.3) is said to *converge*. The notation of $\mathcal{L}(f)$ will be used to denote the Laplace transform of f, and the integral is the ordinary Riemann (improper) integral.[5] The parameter s belongs to some domain on the real line or in the complex plane.

The symbol $\mathcal{L}$ is the *Laplace transformation*, which acts on functions $f = f(t)$ and generates a new function, $F(s) = \mathcal{L}\big[f(t)\big]$ i.e., transforms the function from time domain to frequency domain.

[5]The function f is said to be *Riemann integrable* if there is a number I_{ab} such that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for each partition Δ of $\big[a, b\big]$ with $\|\Delta\| < \delta$, we have

$$\left| \sum_{i=1}^n f(x_i)\big(t_i - t_{i-1}\big) - I_{ab} \right| < \varepsilon \quad ,$$

for all choices of $x_i \in \big[t_{i-1}, t_i\big]$, $i = 1, ..., n$. The value I_{ab} is the *Riemann integral* of f over $[a, b]$ and is written as

$$I_{ab} = \int_a^b f(t)dt \quad .$$

Since the Laplace transform is a linear transform, taking the Laplace transform of equation (4.2) yields,[6]

$$x(s) = \frac{-\pi(t_0)}{(s-c)(s-a)} + \frac{x(t_0)}{s-a} \quad . \tag{4.4}$$

4.2 Inverse of the Laplace Transform

In order to apply the Laplace transform to physical problems, it is necessary to invoke the inverse transform. If $\mathcal{L}\big[f(t)\big] = F(s)$, then the *inverse Laplace transform* is denoted by

$$\mathcal{L}^{-1}\big[F(s)\big] = f(t) \quad , \quad t \geq 0 \quad ,$$

which maps the Laplace transform of a function back to the original function.

Theorem 1 *For a function $F(s)$, the inverse Laplace transform $\mathcal{L}^{-1}\big[F(s)\big]$, if it exists, is unique in the sense that we allow a difference of function values on a set that has zero Lebesgue measure (meaning: a subset of $\mathbb{R}$ that is negligible for integrals).*

This result is known as *Lerch's theorem.* It says that if we restrict our attention to functions that are continuous on $[0,\infty]$, then the inverse transform, $\mathcal{L}^{-1}\big[F(s)\big] = f(t)$, is uniquely defined. Since the functions we are dealing with are solutions to differential equations (hence continuous), the above assumption is completely justified.

The algebraic equation (4.4) is in the frequency domain. We would want to get back to the time domain, as that is what we are interested in. The first term on

[6]The Laplace transform of the first derivative is, $\mathcal{L}\big[\dot{x}(t)\big] = sx(s) - x(0)$, and the Laplace transform of the exponential function is, $\mathcal{L}(e^{ct}) = \dfrac{1}{s-c}$.

the right-hand side of equation (4.4) is the product of the Laplace transform of the individual functions $F(s)$ and $G(s)$. In order to find the inverse transform we exploit the convolution theorem.

4.3 Convolution

The convolution of two functions, $f(t)$ and $g(t)$, defined for $t > 0$, is given by the integral

$$(f * g)(t) = \int_0^t f(T)g(t-T)dT \quad ,$$

which exists if f and g are, say, piecewise continuous. One of the significant properties possessed by the convolution in connection with the Laplace transform is that *the Laplace transform of the convolution of two functions is the product of their Laplace transforms.*

Theorem 2 (Convolution) *If f and g are piecewise continuous on $[0, \infty]$ and of exponential order α, then*

$$\mathcal{L}\big[(f * g)(t)\big] = \mathcal{L}\big[f(t)\big].\mathcal{L}\big[g(t)\big] \quad (Re(s) > \alpha) \quad .$$

Using the convolution theorem in equation (4.4) yields,

$$x(s) = -\pi(t_0)\mathcal{L}(f * g)_t + \frac{x(t_0)}{s-a} \quad . \tag{4.5}$$

Taking the inverse Laplace transform of equation (4.5) yields,[7]

$$x(t) = \mathcal{L}^{-1}\big[x(s)\big] = -\pi(t_0)(f * g)_t + x(t_0)e^{at} \quad . \tag{4.6}$$

Substituting the convolution integral, we get

$$(f * g)_t = \int_{T=0}^{t} f(T)g(t-T)dT = \int_{T=0}^{t} e^{cT}.e^{a(t-T)}dT = \frac{1}{c-a}\big[e^{ct} - e^{at}\big] \quad . \tag{4.7}$$

Substituting equation (4.7) in equation (4.6) gives the general solution for our differential equation,

$$x(t) = \frac{-\pi(t_0)}{c-a}\big[e^{ct} - e^{at}\big] + x(t_0)e^{at} \quad . \tag{4.8}$$

Finally, multiplying throughout by, e^{-at}, and re-arranging gives,

$$x(t_0) = \frac{\pi(t_0)}{c-a}\big[e^{-(a-c)t} - 1\big] + x(t)e^{-at}$$

or

$$P_A(t_0) = \frac{\pi(t_0)}{n-r}[e^{-(r-n)t} - 1] + P_A(t)e^{-rt} \quad . \tag{4.9}$$

The first term on the right-hand side of equation (4.9) is the fundamental value of the patent i.e., the present discounted value of the future stream of profits. The second term on the right-hand side is the difference between the market value of the patent

[7]The inverse transforms of $\mathcal{L}\big[(f * g)(t)\big] = (f * g)(t) = f(t)g(t)$.

$P_A(t_0)$ and its fundamental valuation. As the solution procedure assumes that there are no bubbles, letting $t \to \infty$, we get

$$P_A(t_0) = -\left(\frac{\pi(t_0)}{r-n}\right)\left[\lim_{t\to\infty} e^{-(r-n)t} - 1\right] + \lim_{t\to\infty} P_A(t)e^{-rt} \quad . \tag{4.10}$$

Finally, replacing t by T and t_0 by t in equation (4.10) to get,

$$P_A(t) = \frac{\pi(t)}{r-n} \quad . \tag{4.11}$$

This expression gives us the price of a patent along a balanced growth path.

5 Conclusion

In this note we have shown how to solve for the fundamental value of a patent in continuous time using two methods: the method of integrating factor and the Laplace transform. We have also discussed the criterion for choosing a unique bubble-free solution. The criterion we use, namely the terminal or transversality conditions are widely accepted in practice. The effect of this condition is to ensure a stable path free of extraneous state variables or bubbles.

Declarations

The authors have no relevant financial or non-financial interests to disclose. The authors have no competing interests to declare that are relevant to the content of this article. All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript. The authors have no financial or proprietary interests in any material discussed in this article.

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